ON THE TEIXEIRA CONSTRUCTION OF THE UNICURSAL CUBIC.*

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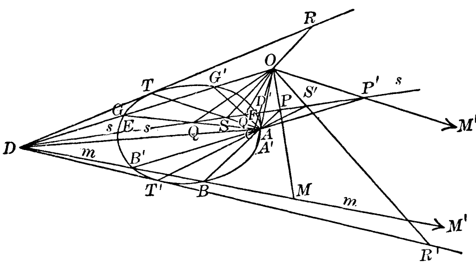
The following is a generalization and a synthetic discussion of the construction of the unicursal cubic due to Teixeira:†

1. Consider a point O, a line s, a conic (C) and two points D and A on s and (C) respectively. A variable line through A meets s in P and (C) again in B. Let $M \equiv (OP, DB)$.

The locus of M is, in general, a unicursal cubic having O for its double point, passing through D and through the points common to s and (C).



Let B' be the second point of intersection of DB with (C), and $P' \equiv (AB', s)$. The point $M' \equiv (OP', DB')$ is clearly another point of the required locus.

When the line $m \equiv DBB'$ turns about the point D describing the pencil of rays (D), the pairs of points B, B' describe an involution on (C), which involution is projected from the fixed point A by an involution of rays (A). The involution of points P, P' on s perspective to (A) is projected from the fixed point O by an involution of rays $O(PP', \cdots)$. We thus have a projective one-to-two correspondence between the pencil (D) and the involution (O). Hence the points M, M' describe a unicursal cubic (C_3) with its double point at O and passing through D ‡ The construction also shows immediately that when m coincides with s, the corresponding points of the locus are the points common to s and (C).

1, *a*. If the line s coincides with the line at infinity, the above construction is identical with Teixeira's.

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† *Nouvelles Annales de Mathématiques*, vol. (1917), pp. 281–284.

‡ Dr. Emil Weyr. Theorie der mehrdeutigen Elementargebilde, etc., p. 13, Leipzig, Teubner, 1869.

1, *b*. It is assumed in the above that the three points O, A, D are distinct, that neither of the points O, A lies on s, nor does the point D lie on (C). Assumptions to the contrary would cause the locus to degenerate or would render the construction meaningless. They are excluded from what follows.

2. Let G, G' be the points of intersection of OD with (C). To the line DO of (D) correspond in (O) the lines projecting from O the traces Q, Q' on s of AG, AG'. Hence: *The lines OQ, OQ' are the tangents to the cubic (C_3) at the double point O.*

2, *a*. The cubic will be crunodal if the line OD cuts (C) in two real points, and acnodal if these points are conjugate imaginary. If O lies on (C) the line OA will be one of the tangents to the cubic at O.

2, *b*. The cubic will be cuspidal, if and only if OD is tangent to (C). The cuspidal tangent joins O to the trace on s of the line AT joining A to the point of contact T of OD with (C). If O coincides with T, the cuspidal tangent will be the line OA.

2, *c*. One of the tangents OQ, OQ' will coincide with OD if the point A coincides with G or G', i. e., when the points O, A, D are collinear. The line OD becomes a united element of the two forms (D) and (O), and the cubic degenerates. This case is excluded from the following considerations.

3. To the ray DA of (D) correspond in (A) the line AD and the tangent a to (C) at A. Let $D' \equiv (as)$. The point of intersection of DA with OD coincides with D, and let $C \equiv (DA, OD')$. Hence: *The line DA is the tangent to the cubic at D. The point C is the tangential of D.*

4. The double elements of the involution (A) are the rays projecting from A the points of contact T, T' of the tangents from D to (C). Let $S \equiv (s, AT)$, $S' \equiv (s, AT')$. The rays OS, OS' are the double elements of the involution (O). The two points of intersection of DT with the cubic (C_3) thus coincide with $R \equiv (DT, OS)$, and those of DT' with (C_3) in $R' \equiv (DT', OS')$. Consequently: *The tangents from D to the conic are also the tangents from D to the cubic, the points of contact with the latter being the points R, R'.*

4, *a*. If the cubic is cuspidal [2, *b*]* one of the tangents from D to (C), and therefore to (C_3), say DT, will coincide with DO, and the point R will coincide with the point O. The two curves will be tangent at T', if the points O, A, T' are collinear.

4, *b*. One of the points R, R', say R, will coincide with D, if and only if the point A coincides with T. Then D is a point of inflection and DA the inflectional tangent. If, in addition, the point O coincides with T', the line $T'A$ is the cuspidal tangent [2, *b*].

* A reference of this sort is to § 2, *b*.

5. Let A' be the second point of intersection of OA with (C). The line OA is one of the two elements of the involution (O), which correspond to the ray DA' of (D). Hence: *The second point common to OA and (C) belongs to the cubic.*

5, *a.* If O lies on (C), the point A' coincides with O, and OA is one of the tangents to the cubic at O (or the cuspidal tangent).

If OA is the tangent to (C) at A, the point A' coincides with A, i. e., A is the tangential of D.

6. We shall now consider the converse proposition. Let (C_3') be a given unicursal cubic and let O denote its double point. Let s be an *arbitrarily chosen straight line,* not passing through O, meeting the cubic in a real point D, and in a pair of points E, F, real, or conjugate imaginary, or coincident, distinct from D. Let A be an *arbitrary* point, distinct from D, on the tangent at D to the cubic. The points Q, Q' being the traces on s of the tangents to (C_3') at the double point O, let $G \equiv (AQ, OD)$, $G' \equiv (AQ', OD)$. The five points A, E, F, G, G' determine a conic (C). This conic, the line s and the points D, O, A, if made to play the same parts, as the similarly named elements in construction [1], will generate a cubic (C_3). The two curves (C_3') and (C_3) will have in common: (i) The double point O [1]; (ii) the tangents OQ, OQ' at the double point [2]; (iii) the three points D, E, F [1]; (iiii) the tangent DA at the point D [3]. Consequently the two cubics are identical.

If the cubic (C_3') has a cusp at O, the conic is to be taken tangent to the line OD at the trace T on OD of the line joining A to the point of intersection of s with the cuspidal tangent. The point T may coincide with O [2, *b*].

If the trace on s of one of the tangents at the double point is taken for the point A, the point O will take the place of one of the two points G, G' in the determination of the conic (C) [5, *a*]; and if the cubic has a cusp at O, the conic is to be taken tangent to OD at O [4, *b*].

The three points D, E, F, will coincide in D, if D is a point of inflection of the given cubic (C_3'), and if s is taken to coincide with the inflectional tangent at D. In the construction [1] the point A is necessarily a point on the tangent at D to the cubic [3], i. e., A in this case has to be a point of s, hence the given cubic cannot be generated by the above construction [1, *b*].

Consequently: *With an arbitrarily chosen straight line an infinite number of conics may be associated in order to generate a given unicursal cubic by the construction [1], provided the line does not pass through the double point of the cubic and is not an inflectional tangent.*

6, *a.* The above discussion solves the problem: *Construct a unicursal*

cubic given: (i) The double point and the two tangents at this point (or the cusp and the cuspidal tangent); (ii) a point D and the tangent at that point; (iii) two points of the cubic collinear with D (or the point of contact of one of the tangents from D to the cubic).

6, *b*. If the line s is taken to coincide with the line at infinity, the restrictions to which s is subjected preclude the possibility of generating, by construction [1], a unicursal cubic having its double point at infinity or having the line at infinity for an inflectional tangent.

7. Given the cubic (C_3') and the line s [6], the conic (C) may also be determined in the following way: Let R, R' be the points of contact of the cubic with the tangents from D to the curve, and let $S \equiv (s, OR)$, $S' \equiv (s, OR')$, $T \equiv (AS, DR)$, $T' \equiv (AS', DR')$. For (C) may be taken the conic which passes through A and is tangent to DR, DR' at the points T, T' respectively [4].

The reader may, referring to the remarks of [4], discuss the special cases, when: (*a*) O is a cusp; (*b*) D is a point of inflection; (*c*) s is one of the tangents from D to the cubic, and the possible combinations of these cases.

The above discussion solves the problem: *Construct a unicursal cubic, given the double point O, a point D, the tangent at this point, and the points of contact R, R', of the tangents from D to the cubic.* (Any line through D may be taken for s.)

8. Let A' be the point of intersection of (C_3') with OA, and C the tangential of D. Let $D' \equiv (s, OC)$. The conic (C) [6] may be taken to pass through A' and to be tangent to AD' at A [5]. These two new conditions may replace in [6] either the two points E, F, or the points G, G', or any two of these four points if they are real. The point A' and the tangent AD' may also replace one of the points R, R' in [7], if these points are real. Finally the elements determining the conic (C) in [6] may be combined with those determining (C) in [7], provided due regard is paid to the reality of these elements. We thus obtain a number of properties of the unicursal cubic and the solution of many construction problems, which properties and problems the reader may find it interesting to formulate.

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